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By Stephen Elmer Slocum.

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By Stephen Elmer Slocum,
Fellow in Mathematics, Clark University, Worcester, Mass.

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§ 1.

The publication of the results of Professor Sophus Lie's investigations in the theory of finite continuous groups, embodied in papers appearing from 1870 to 1898, chiefly in the "Archiv for Mathematik og Naturvidenskab" and the "Forhandlingar i Videnskabs Selskabet i Christiania," and the systematic presentation of his theory in six large volumes, published during the years 1888–1893, opened to mathematicians a new and exceedingly rich field of investigation. As a creator and pioneer in this field, Professor Lie's aim was to outline his theory as broadly as possible, not stopping to obtain entirely rigorous demonstrations of his theorems; and it is not surprising to find that certain of these theorems, and of the fundamental conceptions of his theory, require modification. Thus it appears from a discovery of Study's, mentioned below, that the chief theorem of Lie's theory holds, in general, only in the neighborhood of the identical transformation, and, as a consequence of this fact, that the conception of isomorphism, as developed by Lie, requires modification.

The chief theorem of Lie's theory is that r independent infinitesimal transformations,* whose symbols are

$$X_i \equiv \sum_1^n {}_k\, \xi_{ik}\,(x_1 \ldots x_n)\,\frac{\partial}{\partial x_k}$$
$$(i = 1, 2 \ldots r)$$

(where the ξ's are analytic functions of n independent variables $x_1 \ldots x_n$) generate an r-parameter (*r-gliedrige*) group, in which each transformation

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* Lie terms the symbols of infinitesimal transformation $X_1 \ldots X_r$ *independent* if the ξ's satisfy no linear homogeneous relations of the form

$$e_1\,\xi_{1i}\,(x) + \ldots + e_r\,\xi_{1r}\,(x) \equiv 0$$

simultaneously for $i = 1, 2 \ldots n$, with coefficients e independent of the x's.

is generated by an infinitesimal transformation of the group, if and only if the $X_1 \ldots X_r$ fulfil relations of the form

$$(X_j, X_k) \equiv \sum_1^r {}_s c_{jks} X_s$$
$$(j, k = 1, 2 \ldots r),$$

where (X_j, X_k) denotes the alternant $X_j X_k - X_k X_j$, and the coefficients c_{jks} are quantities independent of the x's.*

In Volume XXXV. of the "Proceedings of the American Academy of Arts and Sciences," pp. 239 *et seq.*, I pointed out an error in the demonstration of what Lie calls the first fundamental theorem,† upon which he bases the demonstration of his chief theorem. This error consists in neglecting conditions imposed at the outset upon certain auxiliary quantities $\mu_1, \mu_2 \ldots \mu_r$, introduced in the course of the demonstration. Thus in the "Continuierliche Gruppen," pp. 372–376 (and substantially in "Transformationsgruppen," III. pp. 558–564), Lie proceeds as follows: Being given at the outset a family with an ∞^r of transformations T_a, defined by the equations

$$x'_i = f_i(x_1, \ldots x_n, a_1, \ldots a_r)$$
$$(i = 1, 2 \ldots n),$$

containing the identical transformation, and such, moreover, that the x''s satisfy a certain system of differential equations, he defines by the introduction of new parameters μ a family of transformations E_μ,

$$x'_i = F_i(\bar{x}'_1, \ldots \bar{x}'_n, \mu_1, \ldots \mu_r)$$
$$(i = 1, 2 \ldots n),$$

each of which is generated by an infinitesimal transformation. Lie then establishes the symbolic equation

$$T_{\bar{a}} E_\mu = T_a,‡$$

* Transformationsgruppen, III. 590; Continuierliche Gruppen, 211, 305, 390.

† Transformationsgruppen, III. 563; Continuierliche Gruppen, 376.

‡ If the equations defining the families of transformations T_a and E_μ are, respectively,

$$x'_i = f_i(x_1 \ldots x_n, a_1 \ldots a_r)$$
$$(i = 1, 2 \ldots n),$$

and

$$x'_i = F_i(\bar{x}'_1 \ldots \bar{x}'_n, \mu_1 \ldots \mu_r)$$
$$(i = 1, 2 \ldots n),$$

the symbolic equation $T_{\bar{a}} E_\mu = T_a$ is equivalent to the simultaneous system of equations

where the $\bar{a}$'s and μ's are arbitrary, and

$$a_k = \Phi_k(\mu_1 \ldots \mu_r, \bar{a}_1 \ldots \bar{a}_r)$$
$$(k = 1, 2 \ldots r),$$

the Φ's being independent functions of the μ's. For

$$\bar{a}_k = a_k^{(0)}$$
$$(k = 1, 2 \ldots r),$$

the transformation $T_{\bar{a}}$ becomes the identical transformation; and therefore we have

$$E_\mu = T_{a^{(0)}} E_\mu = T_a,^*$$

where

$$a_k = \Phi_k(\mu_1 \ldots \mu_r, a_1^{(0)} \ldots a_r^{(0)})$$
$$(k = 1, 2 \ldots r).$$

Thus every transformation of the family E_μ is a transformation of the family T_a. If, conversely, we could show that, for arbitrary values of the a's, every transformation T_a belonged to the family E_μ, it would follow that

$$T_{\bar{a}} T_a = T_a,^\dagger$$

that is to say, we should then have shown that the family of transformations T_a forms a group.

But, although the Φ's are independent functions of the μ's, nevertheless the μ's in certain cases may be infinite for certain systems of values of the a's; and infinite values of the μ's, by their definition, are excluded

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$$\bar{x}'_i = f_i(x_1 \ldots x_n, \bar{a}_1 \ldots \bar{a}_r),$$
$$x'_i = F_i(\bar{x}'_1 \ldots \bar{x}'_n, \mu_1 \ldots \mu_r), \quad (i = 1, 2 \ldots n),$$
$$x'_i = f_i(x_1 \ldots x_n, a_1 \ldots a_r),$$

or, to the functional equations

$$F_i(f_1(x, \bar{a}) \ldots f_n(x, \bar{a}), \mu_1 \ldots \mu_r) = f_i(x_1 \ldots x_n, a_1 \ldots a_r)$$
$$(i = 1, 2 \ldots n).$$

* That is,

$$F_i(\bar{x}'_1 \ldots \bar{x}'_n, \mu_1 \ldots \mu_r) = F_i(f_1(x, a^{(0)}) \ldots f_n(x, a^{(0)}), \mu_1 \ldots \mu_r) = f_i(x_1 \ldots x_n, a_1 \ldots a_r)$$
$$(i = 1, 2 \ldots n),$$

since

$$\bar{x}'_i = f_i(x_1 \ldots x_n, a_1^{(0)} \ldots a_r^{(0)})$$
$$(i = 1, 2 \ldots n).$$

† That is

$$f_i(f_1(x, \bar{a}) \ldots f_n(x, \bar{a}), a_1 \ldots a_r) = f_i(x_1 \ldots x_n, a_1 \ldots a_r)$$
$$(i = 1, 2 \ldots n).$$

at the outset.* We cannot then assume that every transformation T_a belongs to the family E_μ.

We may, however, proceed as follows: For all values of the a's for which the functions $\mu_j = M_j(a_1 \ldots a_r, a_1^{(0)} \ldots a_r^{(0)})$ $(j = 1, 2 \ldots r)$ are finite, we have

$$T_{\bar{a}}\, T_a = T_{\bar{a}}\, E_\mu = T_a,$$

that is,

$$f_i(f_1(x, \bar{a}) \ldots f_n(x, \bar{a}), a_1 \ldots a_r) = F_i(f_1(x, \bar{a}) \ldots f_n(x, \bar{a}), \mu_1 \ldots \mu_r)$$
$$= f_i(x_1 \ldots x_n, a_1 \ldots a_r)$$
$$(i = 1, 2 \ldots n).$$

Let $\beta_1, \beta_2 \ldots$ be a system of values of the a's for which one, or more, of the corresponding μ's is infinite. Also let $b_1, b_2 \ldots$ be the system of values assumed by the a's for $a_k = \beta_k$ $(k = 1, 2 \ldots r)$. Since the functions f are continuous functions of the variables and parameters, and since we assume that the system of parameters β give a definite transformation T_β of the family, we have

$$f_i(f_1(x, \bar{a}) \ldots f_n(x, \bar{a}), \beta_1 \ldots \beta_r)$$
$$= \lim_{a=\beta} f_i(f_1(x, \bar{a}) \ldots f_n(x, \bar{a}), a_1 \ldots a_r)$$
$$= \lim_{a=b} f_i(x_1 \ldots x_n, a_1 \ldots a_r) = f_i(x_1 \ldots x_n, b_1 \ldots b_r)$$
$$(i = 1, 2 \ldots n),$$

which is equivalent to the symbolic equation

$$T_{\bar{a}}\, T_\beta = T_{\bar{a}} \lim_{a=\beta} T_a = \lim_{a=\beta} T_{\bar{a}}\, T_a = \lim_{a=b} T_a = T_b.$$

Consequently, the composition of two arbitrary transformations $T_{\bar{a}}$ and T_β of the family is equivalent to a transformation T_b of this family; that is to say, the family of transformations T_a forms a group. The transformation T_b, however, may not be a transformation of the group that can be generated by an infinitesimal transformation of the group. Thus, every transformation of a group with continuous parameters and containing the identical transformation is not necessarily generated by an infinitesimal transformation of the group.†

Professors Study and Engel were the first to point this out, and thus establish a distinction between a group with continuous parameters and a continuous group.‡ They found that not every transformation of the

* These Proceedings, XXXV. 247.

† These Proceedings, XXXV. 483–485.

‡ Engel: Leipziger Berichte, 1892.

special linear homogeneous group can be generated by an infinitesimal transformation of the group, and consequently this group is not properly continuous in the sense in which Lie uses the term. Since this important discovery, the subject of continuity has been investigated for the case of the general linear homogeneous group and its sub-groups, as well as for various other groups, by Professor Taber and his pupils, and from a geometrical standpoint by Professors Newson and Emch.*

This paper contains an investigation of the relation of the continuity of a group generated by infinitesimal transformations to its structure (*Zusammensetzung*), and the classification of all possible types of structure of complex groups with two, three, and four parameters with reference to the continuity of groups of these types. All possible types of groups with two, three, and four parameters can be divided into three classes. Every group is continuous whose structure is of a type belonging to the first class; every group is discontinuous whose structure is of a type drawn from the second class; and of the groups whose structure is of a type belonging to the third class, some are continuous and some are discontinuous. The parameter group of a given group G_r has the same structure as G_r; and every group of a given structure has the same parameter group. In every case which I have examined, the parameter group is discontinuous unless its type of structure is of the first class. I have considered not only complex groups, but also real groups generated by infinitesimal transformations.

§ 2.

The criterion for the continuity of an r-parameter group G_r is obtained as follows. Let $X_1 \ldots X_r$ be any system of independent infinitesimal transformations of G_r. The equations of G_r in their *canonical form* † are then

$$(1) \qquad x'_i = f_i(x_1 \ldots x_n, a_1 \ldots a_r) \qquad (i = 1, 2 \ldots n),$$

where $f_i(x, a)$, for $i = 1, 2 \ldots n$, is defined in the neighborhood of the identical transformation by the series

* Taber: Am. Jour. Maths., XVI.; Bull. Am. Math. Soc., July, 1894, April, 1896, Jan. 1897, Feb. 1900; Math. Ann., XLVI.; These Proceedings, XXXV. 577. Rettger: These Proceedings, XXXIII. 493–499; Am. Jour. Maths., XXII. Williams: These Proceedings, XXXV. 97–107. Newson: Kansas Univ. Quart., IV., V. 1896. Emch: Kansas Univ. Quart., IV., V. 1896.

† Transformationsgruppen, I. 171, III. 607; Continuierliche Gruppen, 454.

$$x_i + \sum_1^r{}_j a_j X_j x_i + \frac{1}{2!}\sum_1^r{}_j \sum_1^r{}_k a_j a_k X_j X_k x_i + \ldots$$

The transformation defined by equations (1) (the general transformation of this group) may be denoted by T_a. For finite values of the parameters $a_1, a_2 \ldots a_r$, the transformation T_a is generated by the infinitesimal transformation

$$a_1 X_1 + a_2 X_2 + \ldots + a_r X_r;$$

but for infinite values of $a_1, a_2 \ldots a_r$, T_a is not generated by an infinitesimal transformation of the group unless $T_a = T_\alpha$, the parameters $\alpha_1, \alpha_2 \ldots \alpha_r$ being all finite.* The transformation T_b is defined by

(2) $$x''_i = f_i(x'_1 \ldots x'_n, b_1 \ldots b_r) \quad (i = 1, 2 \ldots n);$$

and the transformation $T_b T_a$, obtained by the composition of the transformations T_a and T_b,† is equivalent to a transformation T_c, defined by

(3) $$x''_i = f_i(x_1 \ldots x_n, c_1 \ldots c_r) \quad (i = 1, 2 \ldots n),$$

where

(4) $$c_k = \varphi_k(a_1 \ldots a_r, b_1 \ldots b_r) \quad (k = 1, 2 \ldots r).$$

If the c's can be taken finite for every finite system of values of the a's and b's, the group is continuous. If, however, it is possible to assign finite values to the a's and b's such that in each system of values of the c's one (or more) of the c's becomes infinite, the transformation $T_b T_a$ cannot be generated by an infinitesimal transformation of the group, and consequently the group is discontinuous.‡ A transformation which cannot be generated by an infinitesimal transformation of the group may be termed *essentially singular.*§ If the parameters a and b are taken sufficiently small, the transformation $T_b T_a$ can always be generated by an infinitesimal transformation, and, consequently, Lie's chief theorem holds in the neighborhood of the identical transformation.

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* Taber: These Proceedings, XXXV. 579.

† $T_b T_a$ will denote the transformation obtained by applying to the manifold $(x_1 \ldots x_n)$ first the transformation T_a and then the transformation T_b. Lie denotes this resultant transformation by $T_a T_b$.

‡ Cf. Rettger: Am. Jour. Maths., XXII.

§ Taber: Bull. Am. Math. Soc., VI. 199–203; These Proceedings, XXXV. 580.

If the system of equations (4) be written in the form

$$(5) \qquad a'_k = \phi_k(a_1 \ldots a_r, \alpha_1 \ldots \alpha_r) \quad (k = 1, 2 \ldots r),$$

it can be shown that they define an r-parameter group in the variables a and a', with parameters $\alpha_1 \ldots \alpha_r$. That is to say, from

$$(5) \qquad a'_k = \phi_k(a_1 \ldots a_r, \alpha_1 \ldots \alpha_r) \quad (k = 1, 2 \ldots r)$$

and

$$(6) \qquad a''_k = \phi_k(a'_1 \ldots a'_r, \beta_1 \ldots \beta_r) \quad (k = 1, 2 \ldots r),$$

we have

$$(7) \qquad a''_k = \phi_k(a_1 \ldots a_r, \gamma_1 \ldots \gamma_r) \quad (k = 1, 2 \ldots r),$$

where

$$(8) \qquad \gamma_j = \phi_j(\alpha_1 \ldots \alpha_r, \beta_1 \ldots \beta_r) \quad (j = 1, 2 \ldots r).$$

The group thus defined is termed the *parameter group* of the group G_r.* Since the equations defining the transformations of the parameter group involve the functions ϕ, this group is especially important in the study of groups generated by infinitesimal transformations.

In general there is more than one system of functions ϕ such that

$$T_c = T_b T_a,$$

provided

$$c_j = \phi_j(a_1 \ldots a_r, b_1 \ldots b_r) \quad (j = 1, 2 \ldots r).$$

But it may happen that the equations defining one group of a given structure restrict the functions c to fewer systems of values than in the case of another group of the same structure. Thus it is possible that of two groups of a given structure one shall be continuous and the other discontinuous.†

These statements are exemplified by a consideration of two groups $G_2^{(1)}$ and $G_2^{(2)}$, whose infinitesimal transformations are, respectively, p_1, $x_1 p_1$, and p_2, $x_2 p_2 + p_1$.‡ Both of these groups have the structure

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* Transformationsgruppen, I. 401 *et seq.*

† Cf. Bull. Am. Math. Soc., VI. 202.

‡ Throughout this paper Lie's notation will be followed, in accordance with which

$$p_1 \equiv \frac{\partial}{\partial x_1}, \quad p_2 \equiv \frac{\partial}{\partial x_2}, \quad \ldots \quad p_r \equiv \frac{\partial}{\partial x_r}.$$

$(X_1, X_2) \equiv X_1$. The canonical form of the finite equations of the group p_1, $x_1 p_1$ is

$$
(9) \qquad \begin{aligned} x'_1 &= x_1 e^{a_2} + \frac{a_1}{a_2}(e^{a_2} - 1), \\ x'_2 &= x_2. \end{aligned}
$$

These equations define the transformation T_a of $G_2^{(1)}$. Similarly, the equations defining the transformation T_b of $G_2^{(1)}$ are

$$
(10) \qquad \begin{aligned} x''_1 &= x'_1 e^{b_2} + \frac{b_1}{b_2}(e^{b_2} - 1), \\ x''_2 &= x'_2. \end{aligned}
$$

The transformation $T_b T_a$, obtained by the composition of the transformations T_a and T_b, is defined by

$$
(11) \qquad \begin{aligned} x''_1 &= e^{a_2 + b_2} x_1 + \frac{a_1}{a_2}(e^{a_2} - 1)\, e^{b_2} + \frac{b_1}{b_2}(e^{b_2} - 1), \\ x''_2 &= x_2; \end{aligned}
$$

and, if this is equivalent to a transformation T_c of the group, we have also

$$
(12) \qquad \begin{aligned} x''_1 &= x_1 e^{c_2} + \frac{c_1}{c_2}(e^{c_2} - 1), \\ x''_2 &= x_2. \end{aligned}
$$

Therefore

$$
(13) \qquad \begin{aligned} c_1 &= \frac{a_2 + b_2 + 2 k \pi \sqrt{-1}}{e^{a_2 + b_2} - 1}\left[e^{b_2} \frac{a_1}{a_2}(e^{a_2} - 1) + \frac{b_1}{b_2}(e^{b_2} - 1)\right] \\ &\equiv \phi_1(a_1, a_2, b_1, b_2), \\ c_2 &= a_2 + b_2 + 2 k \pi \sqrt{-1}, \qquad \equiv \phi_2(a_1, a_2, b_1, b_2), \end{aligned}
$$

where k is an arbitrary integer. Consequently for $G_2^{(1)}$ there is more than one system of functions ϕ. Provided $a_2 + b_2$ is not an even multiple of $\pi\sqrt{-1}$, every system of values of c_1 and c_2 is finite. For $a_2 + b_2 = 2 \kappa \pi \sqrt{-1}$, c_2 is finite, but the denominator of c_1 becomes zero. In this case, however, that system of values of c_1 corresponding to $k = -\kappa$ is finite. Consequently the parameters c_1 and c_2 can always be chosen finite, and therefore $G_2^{(1)}$ is continuous.

For the group $G_2^{(2)}$, whose infinitesimal transformations are p_2, $x_2 p_2 + p_1$, T_a is defined by

$$
(14) \qquad \begin{aligned} x'_1 &= x_1 + a_2, \\ x'_2 &= x_2 e^{a_2} + \frac{a_1}{a_2}(e^{a_2} - 1), \end{aligned}
$$

and T_b by

$$(15)\qquad \begin{aligned} x''_1 &= x'_1 + b_2, \\ x''_2 &= x'_2 e^{b_2} + \frac{b_1}{b_2}(e^{b_2} - 1). \end{aligned}$$

Consequently the transformation $T_b\, T_a$ is defined by

$$(16)\qquad \begin{aligned} x''_1 &= x_1 + a_2 + b_2, \\ x''_2 &= x_2 e^{a_2 + b_2} + e^{b_2}\frac{a_1}{a_2}(e^{a_2} - 1) + \frac{b_1}{b_2}(e^{b_2} - 1), \end{aligned}$$

whence, if $T_b\, T_a = T_c$,

$$(17)\qquad \begin{aligned} c_1 &= \frac{a_2 + b_2}{e^{a_2 + b_2} - 1}\left[e^{b_2}\frac{a_1}{a_2}(e^{a_2} - 1) + \frac{b_1}{b_2}(e^{b_2} - 1)\right] \equiv \phi_1(a_1, a_2, b_1, b_2), \\ c_2 &= a_2 + b_2. \qquad\qquad \equiv \phi_2(a_1, a_2, b_1, b_2). \end{aligned}$$

In this case there is but one system of functions ϕ. If, now, $a_2 + b_2$ is an even multiple of $\pi\sqrt{-1}$, c_2 is finite, but c_1 is infinite; that is, there is no finite parameter c_1 corresponding to this choice of the parameters a and b. Consequently, if $a_2 + b_2 = 2\,k\,\pi\sqrt{-1} \neq 0$, $T_b\, T_a$ cannot be generated by an infinitesimal transformation of the group, and therefore $G_2^{(2)}$ is discontinuous.*

Lie states that two groups having the same structure are (holohedrically) isomorphic; but the groups $G_2^{(1)}$ and $G_2^{(2)}$ are not properly isomorphic, except in the neighborhood of the identical transformation, since one is continuous and the other discontinuous. Whence it appears that the conception of isomorphism, as developed by Lie, requires modification.

The parameter group of $G_2^{(1)}$ is defined by the equations

$$(13\text{ a})\qquad \begin{aligned} a'_1 &= \frac{a_2 + \alpha_2 + 2\,k\,\pi\sqrt{-1}}{e^{a_2 + \alpha_2} - 1}\left[e^{\alpha_2}\frac{a_1}{a_2}(e^{a_2} - 1) + \frac{\alpha_1}{\alpha_2}(e^{\alpha_2} - 1)\right], \\ a'_2 &= a_2 + \alpha_2 + 2\,k\,\pi\sqrt{-1}, \end{aligned}$$

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* For the group $\frac{x_2 p_1}{2}$, $\frac{1}{2}(x_1 p_1 - x_2 p_2)$, which also has the structure $(X_1, X_2) \equiv X_1$, we have

$$\begin{aligned} c_1 &= \frac{a_2 + b_2 + 4\,k\,\pi\sqrt{-1}}{e^{a_2 + b_2} - 1}\left[e^{b_2}\frac{a_1}{a_2}(e^{a_2} - 1) + \frac{b_1}{b_2}(e^{b_2} - 1)\right] \equiv \phi_1(a_1, a_2, b_1, b_2), \\ c_2 &= a_2 + b_2 + 4\,k\,\pi\sqrt{-1}; \qquad\qquad \equiv \phi_2(a_1, a_2, b_1, b_2), \end{aligned}$$

and if $a_2 + b_2 = 2(2\,\kappa + 1)\,\pi\sqrt{-1}$, where κ is an integer, c_1 is always infinite. Consequently this group is also discontinuous.

where k is any integer, and of $G_2^{(2)}$ by the equations

$$\text{(17 a)} \quad \begin{aligned} a'_1 &= \frac{a_2 + \alpha_2}{e^{a_2 + \alpha_2} - 1} \left[e^{\alpha_2} \frac{a_1}{a_2} (e^{a_2} - 1) + \frac{\alpha_1}{\alpha_2} (e^{\alpha_2} - 1) \right], \\ a'_2 &= a_2 + \alpha_2. \end{aligned}$$

Nevertheless, for any value of k, the parameter group of $G_2^{(1)}$ is identical with the parameter group of $G_2^{(2)}$. For let $S_a^{(1)}$ denote the transformation defined by equations (13 a), and $S_a^{(2)}$ the transformation defined by equations (17 a). Then for any system of values of α_1, α_2, and for any value of k, we have, by properly choosing β_1, β_2,

$$S_a^{(1)} = S_\beta^{(2)};$$

which symbolic equation persists for β_1, β_2, and k arbitrary, if α_1, α_2 are properly chosen.*

Let T_a be an arbitrary transformation of G_r, defined by the equations

$$x'_i = x_i + \sum_1^r {}_j a_j X_j x_i + \frac{1}{2!} \sum_1^r {}_j \sum_1^r {}_k a_j a_k X_j X_k x_i + \ldots$$
$$(i = 1, 2 \ldots n).$$

The transformation T_a^{-1}, inverse to T_a, is then defined by the equations which we obtain on replacing a_1, $a_2 \ldots$ by their negatives.† Compound the transformation T_α and its inverse with each transformation T_a of G_r so as to obtain the transformation $T_\alpha T_a T_\alpha^{-1}$, which is also a transformation of G_r; and let

$$\text{(18)} \quad T_{a'} = T_\alpha T_a T_\alpha^{-1}. \ddagger$$

Let

$$\text{(19)} \quad T_{a''} = T_\beta T_{a'} T_\beta^{-1}.$$

Then, if

$$\text{(20)} \quad T_\gamma = T_\beta T_\alpha,$$

we have

$$\text{(21)} \quad T_{a''} = T_\gamma T_a T_\gamma^{-1}.$$

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* Equations (13 a) may be regarded as defining a group with three parameters a_1, a_2, and k, of which two, a_1 and a_2, vary continuously, and one, namely k, takes only integer values. But this group is not a mixed group, since we have shown that k is unessential, that is, it is immaterial what value is assigned to k.

† Transformationsgruppen, I. 52, 53.

‡ The transformation $T_{a'}$ is said by Lie to be obtained by the application (*Ausführung*) of T_α to the transformations T_a of G_r. Cf. Lie: Continuierliche Gruppen, 445 *et seq.*

The symbolic equation (18) may be regarded as defining a transformation between the parameters a and a' of G_r, and is equivalent to r equations of the form

$$a'_j = F_j(a_1 \ldots a_r, \alpha_1 \ldots \alpha_r) \quad (j = 1, 2 \ldots r). \tag{22}$$

Similarly, (19) is equivalent to

$$a''_j = F_j(a'_1 \ldots a'_r, \beta_1 \ldots \beta_r) \quad (j = 1, 2 \ldots r),$$

and (21) to

$$a''_j = F_j(a_1 \ldots a_r, \gamma_1 \ldots \gamma_r) \quad (j = 1, 2 \ldots r),$$

and, in virtue of (20),

$$\gamma_j = \phi_j(\alpha_1 \ldots \alpha_r, \beta_1 \ldots \beta_r) \quad (j = 1, 2 \ldots r).$$

Thus equations (22) define a group Γ, which is termed the *adjoined* of G_r.* The number of variables of the group Γ is r, and it contains r parameters, but these are not necessarily all essential. The number of essential parameters in Γ is less than r by one for each independent infinitesimal transformation of G_r commutative with each of the infinitesimal transformations $X_1 \ldots X_r$.† Thus, if G_r contains just s such independent infinitesimal transformations, Γ is an $(r - s)$-parameter group.

The canonical form of the equations defining the transformation T_a of $G_2^{(1)}$ is

$$\begin{aligned} x'_1 &= x_1 e^{a_2} + \frac{a_1}{a_2}(e^{a_2} - 1), \\ x'_2 &= x_2, \end{aligned} \tag{9 a}$$

and consequently, if $T_{a'} = T_\alpha T_a T_\alpha^{-1}$, we have

$$\begin{aligned} a'_1 &= \frac{\alpha_2 + 2k\pi\sqrt{-1}}{a_2} a_1 e^{-\alpha_2} - (a_2 + 2k\pi\sqrt{-1})\frac{\alpha_1}{\alpha_2}(e^{-\alpha_2} - 1) \\ &\equiv F_1(a_1, a_2, \alpha_1, \alpha_2), \\ a'_2 &= a_2 + 2k\pi\sqrt{-1} \equiv F_2(a_1, a_2, \alpha_1, \alpha_2), \end{aligned} \tag{23}$$

where k is an arbitrary integer. The family of transformations between the variables a and a' which we obtain for any assigned integer value of

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* Transformationsgruppen, I. 272, 275; III. 667–670. Continuierliche Gruppen, 454–455.

† Transformationsgruppen, I. 277.

k interchanges the transformations of $G_2^{(1)}$ (so that T_a becomes $T_{a'}$), but this family of transformations does not form a group, except for $k = 0$; in which case it is the adjoined of $G_2^{(1)}$. This adjoined group, $\Gamma^{(1)}$, is generated by the infinitesimal transformations

$$-a_2\frac{\partial}{\partial a_1}, \quad a_1\frac{\partial}{\partial a_1}.$$

We may regard a_1, a_2, and k as parameters, a_1, a_2 varying continuously, and k taking only integer values, and then we have a family of transformations (interchanging the transformations of $G_2^{(1)}$) that forms a mixed group, of which $\Gamma^{(1)}$ is a sub-group. Only those transformations of this mixed group which belong to $\Gamma^{(1)}$ are generated by an infinitesimal transformation of this mixed group. This mixed group might be called the adjoined of $G_2^{(1)}$, in which case the adjoined of a given group G_r would appear as a mixed group containing more than r parameters, some of which, however, do not vary continuously.

In the case of the group $G_2^{(2)}$ the transformation T_a is defined by the equations

$$\text{(14 a)} \qquad \begin{aligned} x'_1 &= x_1 + a_2, \\ x'_2 &= x_2 e^{a_2} + \frac{a_1}{a_2}(e^{a_2} - 1), \end{aligned}$$

and if $T_{a'} = T_{\alpha} T_a T_{\alpha}^{-1}$, we have

$$\text{(24)} \qquad \begin{aligned} a'_1 &= a_1 e^{-\alpha_2} - a_2\frac{\alpha_1}{\alpha_2}(e^{-\alpha_2} - 1) \equiv F_1(a_1, a_2, \alpha_1, \alpha_2), \\ a'_2 &= a_2 \qquad\qquad\qquad\qquad\quad \equiv F_2(a_1, a_2, \alpha_1, \alpha_2). \end{aligned}$$

Consequently the adjoined of the group $G_2^{(2)}$ cannot be regarded as a mixed group. Thus the equations of the adjoined, obtained from the symbolic equation $T_{a'} = T_{\alpha} T_a T_{\alpha}^{-1}$, are not necessarily all linear and homogeneous. However, they will always include one system of linear homogeneous equations that define a family of transformations generated by infinitesimal transformations, and forming a group.

Lie has shown that if $X_1 \ldots X_r$ generate an r-parameter group G_r in the n variables $x_1 \ldots x_n$, and subject to the conditions

$$(X_j, X_k) \equiv \sum_1^r{}_s c_{jks} X_s$$
$$(j, k = 1, 2 \ldots r),$$

the c's being the structural constants, the adjoined group is generated by the infinitesimal transformations

$$E_\nu \equiv \sum_1^r {}_k \sum_1^r {}_\mu c_{\mu\nu k} a_\mu \frac{\partial}{\partial a_k}$$

$$(\nu = 1, 2 \ldots r),$$

and $E_1 \ldots E_r$ satisfy the conditions

$$(E_j, E_k) \equiv \sum_1^r {}_s c_{jks} E_s$$

$$(j, k = 1, 2 \ldots r).*$$

The infinitesimal transformations $E_1 \ldots E_r$, however, are not necessarily all independent. The number of independent infinitesimal transformations of the adjoined of G_r will be one less for each infinitesimal transformation of G_r that is commutative with every infinitesimal transformation of G_r (*ausgezeichnete Transformation*), as mentioned above, page 95. Such a transformation will be called an extraordinary transformation of G_r. It follows, from what has been said, that every group of the same structure has the same adjoined. If G_r contains no extraordinary transformation, G_r and Γ have the same structure. If Γ contains an essentially singular transformation, G_r must also contain at least one essentially singular transformation. Therefore, if Γ is discontinuous, every group of which Γ is the adjoined is discontinuous.† But Γ is not necessarily discontinuous if G_r contains an essentially singular transformation.

By Lie's theorem,‡ the infinitesimal transformations of the adjoined of $G_2^{(1)}$ and also of $G_2^{(3)}$ (since both have the same structure) are

$$-a_2 \frac{\partial}{\partial a_1}, \quad a_1 \frac{\partial}{\partial a_1},$$

and thus the finite equations of the adjoined are

$$(25) \qquad \begin{aligned} a'_1 &= a_1 e^{a_1} - a_2 \frac{a_1}{a_2} (e^{a_1} - 1), \\ a'_2 &= a_2, \end{aligned}$$

which result agrees with the equations deduced page 95.

§ 3.

In what follows I shall denote by α, β, γ, respectively, the following differential operators:

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* Transformationsgruppen, I. 275; Continuierliche Gruppen, pp. 466–467.

† Taber: Bull. Am. Math. Soc., VI. 203; These Proceedings, XXXV. 590.

‡ Cf. Continuierliche Gruppen, p. 467.

$$\alpha = a_1 X_1 + a_2 X_2 + \ldots + a_r X_r,$$
$$\beta = b_1 X_1 + b_2 X_2 + \ldots + b_r X_r,$$
$$\gamma = c_1 X_1 + c_2 X_2 + \ldots + c_r X_r,$$

where the a's, b's, and c's denote arbitrary parameters, and by e^{α} the operator

$$(1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \ldots)f \equiv f + \alpha f + \frac{\alpha^2}{2!}f + \frac{\alpha^3}{3!}f + \ldots,$$

where $\alpha^{m+1}f \equiv \alpha(\alpha^m f)$.

Let

$$X = \sum_1^n {}_i\, \xi_i(x) \frac{\partial}{\partial x_i},$$

and let

$$x'_i = x_i + t X x_i + \frac{t^2}{2!} X^2 x_i + \frac{t^3}{3!} X^3 x_i + \ldots = e^{tX} x_i$$
$$(i = 1, 2 \ldots n).$$

Since the x''s are functions of t, any function of the x''s, $f(x'_1 \ldots x'_n)$, is also a function of t. And we have

$$f(x'_1 \ldots x'_n)$$
$$= [f(x')]_{t=0} + t\left[\frac{df(x')}{dt}\right]_{t=0} + \frac{t^2}{2!}\left[\frac{d^2f(x')}{dt^2}\right]_{t=0} + \frac{t^3}{3!}\left[\frac{d^3f(x')}{dt^3}\right]_{t=0} + \ldots$$
$$(i = 1, 2 \ldots n).$$

But

$$\frac{df(x')}{dt} = \sum_1^n {}_i \frac{\partial f(x')}{\partial x'_i}\frac{dx'_i}{dt} = \sum_1^n {}_i\, \xi_i(x')\frac{\partial f(x')}{\partial x'_i} = X' f(x'),$$

where X' denotes the result of substituting the accented for the unaccented variables in X. Therefore

$$f(x'_1 \ldots x'_n)$$
$$= [f(x')]_{t=0} + t[X'f(x')]_{t=0} + \frac{t^2}{2!}[X'^2 f(x')]_{t=0} + \frac{t^3}{3!}[X'^3 f(x')]_{t=0} + \ldots$$
$$= f(x) + t X f(x) + \frac{t^2}{2!} X^2 f(x) + \frac{t^3}{3!} X^3 f(x) + \ldots$$
$$= e^{tX} f(x_1 \ldots x_n).$$

Consequently, if

$$x'_i = e^{\alpha} x_i$$
$$(i = 1, 2 \ldots n),$$

we have

$$f_i(x'_1 \ldots x'_n) = e^{\alpha} f_i(x_1 \ldots x_n)$$
$$(i = 1, 2 \ldots n).$$

Let now

$$x'_i = f_i(x_1 \ldots x_n, a_1 \ldots a_r) = e^{\alpha} x_i$$
$$(i = 1, 2 \ldots n),$$

and

$$x''_i = f_i(x'_1 \ldots x'_n, b_1 \ldots b_r) = e^{\beta'} x'_i$$
$$(i = 1, 2 \ldots n),$$

where β' denotes the result of substituting the accented for the unaccented variables in the X's which appear in the operator β. Then, by what precedes, we have

$$\begin{aligned} x''_i &= f_i(x'_1 \ldots x'_n, b_1 \ldots b_r) \\ &= e^{\alpha} f_i(x_1 \ldots x_n, b_1 \ldots b_r) = e^{\alpha}(e^{\beta} x_i) \end{aligned}$$
$$(i = 1, 2 \ldots n).$$

Let the operator $e^{\alpha} e^{\beta}$ be defined as follows:

$$(e^{\alpha} e^{\beta}) f$$
$$= (1 + (\alpha + \beta) + \frac{1}{2!}(\alpha^2 + 2\alpha\beta + \beta^2) + \frac{1}{3!}(\alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3) + \ldots) f$$
$$= f + (\alpha + \beta) f + \frac{1}{2!}(\alpha^2 + 2\alpha\beta + \beta^2) f + \frac{1}{3!}(\alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3) f + \ldots$$

Then

$$(e^{\alpha} e^{\beta}) x_i = e^{\alpha}(e^{\beta} x_i)$$
$$(i = 1, 2 \ldots n),$$

and therefore

$$x''_i = (e^{\alpha} e^{\beta}) x_i$$
$$(i = 1, 2 \ldots n);$$

thus $e^{\alpha} e^{\beta}$ denotes the result of the composition in the order named of the transformations denoted by e^{α} and e^{β}.*

By § 2, page 94, the transformation inverse to e^{α} is $e^{-\alpha}$. Let δt denote an infinitesimal constant. Since the transformation $e^{\alpha + \delta t \gamma}$ is infinitely near the transformation e^{α}, the transformation $e^{-\alpha} e^{\alpha + \delta t \gamma}$ is an infinitesimal transformation. If we denote its parameters by $\delta t\, b_1$, $\delta t\, b_2 \ldots \delta t\, b_r$, we have

$$e^{-\alpha} e^{\alpha + \delta t \gamma}$$
$$= 1 + \delta t \{\gamma - \frac{1}{2!}(\alpha, \gamma) + \frac{1}{3!}(\alpha, (\alpha, \gamma)) - \frac{1}{4!}(\alpha, (\alpha, (\alpha, \gamma))) + \ldots\} + \ldots$$
$$= e^{\delta t \beta} = 1 + \delta t \beta + \ldots$$

in which (α, γ) denotes the alternant $\alpha\gamma - \gamma\alpha$; and neglecting infinitesimals of the second and higher orders, we have

$$\beta = \gamma - \frac{1}{2!}(\alpha, \gamma) + \frac{1}{3!}(\alpha, (\alpha, \gamma)) - \frac{1}{4!}(\alpha, (\alpha, (\alpha, \gamma))) + \ldots \tag{26}$$

* Cf. Campbell: Proc. London Math. Soc., XXVIII. 381–390. Also Poincaré: Comptes Rendus, Mai 1er, 1899.

By supposition $\gamma = \overset{r}{\underset{1}{\Sigma}}_j c_j X_j$; and since the r infinitesimal transformations $X_1 \ldots X_r$ satisfy the Lieschen criterion,

$$(\alpha, \gamma) = (\overset{r}{\underset{1}{\Sigma}}_j a_j X_j, \overset{r}{\underset{1}{\Sigma}}_k c_k X_k) = \overset{r}{\underset{1}{\Sigma}}_j [(a_1 c_2 - a_2 c_1) c_{12j} + (a_1 c_3 - a_3 c_1) c_{13j} + \ldots] X_j.$$

Whence it follows that $(\alpha, (\alpha, \gamma))$, etc., are linear in $X_1 \ldots X_r$. It is to be observed that each term in the right member of (26) is linear in $c_1 \ldots c_r$. Since $X_1 \ldots X_r$ are independent, the coefficients of corresponding X's in the two members of (26) are equal. Therefore

$$(27) \quad \begin{aligned} b_1 &= G_{11} c_1 + G_{12} c_2 + \ldots + G_{1r} c_r, \\ b_2 &= G_{21} c_1 + G_{22} c_2 + \ldots + G_{2r} c_r, \\ &\ldots\ldots\ldots\ldots \\ b_r &= G_{r1} c_1 + G_{r2} c_2 + \ldots + G_{rr} c_r, \end{aligned}$$

in which the G's are integral functions of $a_1 \ldots a_r$.*

Let the determinant of the G's be denoted by Δ, that is, let

$$\Delta \equiv \begin{vmatrix} G_{11}, & G_{12} & \ldots & G_{1r} \\ G_{21}, & G_{22} & \ldots & G_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ G_{r1}, & G_{r2} & \ldots & G_{rr} \end{vmatrix}$$

The symbolic equation

$$e^{-\alpha} e^{\alpha + \delta t \gamma} = e^{\delta t \beta}$$

may be written

$$e^{\alpha} e^{\delta t \beta} = e^{\alpha + \delta t \gamma}.$$

If $\Delta \neq 0$ we may take $b_1 \ldots b_r$ arbitrarily, and by means of equations (27) determine the c's to satisfy this symbolic equation; in which case

$$(28) \quad \begin{aligned} c_1 &= \frac{A_{11}}{\Delta} b_1 + \frac{A_{12}}{\Delta} b_2 + \ldots + \frac{A_{1r}}{\Delta} b_r, \\ c_2 &= \frac{A_{21}}{\Delta} b_1 + \frac{A_{22}}{\Delta} b_2 + \ldots + \frac{A_{2r}}{\Delta} b_r, \\ &\ldots\ldots\ldots\ldots \\ c_r &= \frac{A_{r1}}{\Delta} b_1 + \frac{A_{r2}}{\Delta} b_2 + \ldots + \frac{A_{rr}}{\Delta} b_r, \end{aligned}$$

* Schur and Engel. See Transformationsgruppen, III. 754 *et seq.* and 788 *et seq* See also These Proceedings, XXXV. 584, 585.

where $A_{\mu\nu}$ is the first minor of Δ relative to $G_{\nu\mu}$, and thus the A's are integral functions of $a_1, a_2 \ldots a_r$. Consequently, if $\Delta \neq 0$, the composition of the transformation e^{a} and an arbitrary infinitesimal transformation $e^{\delta t\beta}$ gives a transformation $e^{a+\delta t\gamma}$, infinitely near the transformation e^{a}, and generated by an infinitesimal transformation. Let $e^{a+\delta t\gamma}$ be denoted by e^{a_1}, that is, let $e^{a_1} = e^{a+\delta t\gamma}$, where

$$a_1 = a + \delta t \gamma = a_1^{(1)} X_1 + a_2^{(1)} X_2 + \ldots + a_r^{(1)} X_r.$$

Applying the infinitesimal transformation $e^{\delta t\beta}$ repeatedly, we thus obtain the equations

$$\begin{aligned} e^{a_1} &= e^{a}\, e^{\delta t\beta}, \\ e^{a_2} &= e^{a_1} e^{\delta t\beta} = e^{a}\, e^{2\delta t\beta}, \\ e^{a_3} &= e^{a_2} e^{\delta t\beta} = e^{a}\, e^{3\delta t\beta}, \\ &\ldots\ldots\ldots\ldots \\ e^{a_n} &= e^{a_{n-1}} e^{\delta t\beta} = e^{a}\, e^{n\delta t\beta}. \end{aligned}$$

For n infinite, $n\delta t$ is finite, and may be taken equal to unity; thus

$$e^{a_n} = e^{a}\, e^{\beta}.$$

Consequently, if Δ does not vanish for any system of values of $a_1 \ldots a_r$, in which case Δ is a constant,* then the composition of an arbitrary transformation e^{a} with finite parameters with an arbitrary transformation $e^{n\delta t\beta} = e^{\beta}$ with finite parameters, gives a transformation of the group with finite parameters which is generated by an infinitesimal transformation.

The form of Δ depends only on the structural constants, and thus Δ is the same for all groups of the same structure. Therefore, if the Δ corresponding to a given structure is a constant, the composition of two arbitrary transformations of any group of this structure gives a transformation of the group with finite parameters, that is, a non-singular transformation of the group, and consequently every group of this structure is continuous.†

If the Δ corresponding to a given structure vanishes for certain systems of values of $a_1 \ldots a_r$, some groups of this structure may be continuous and others discontinuous. For example, the two groups $G_2^{(1)}$ and $G_2^{(2)}$, considered above, page 92, both have the structure $(X_1, X_2) \equiv X_1$. The

* For complex groups Δ is either unity, or else vanishes for certain systems of values of $a_1 \ldots a_r$. See the expression for Δ as a product on page 104.

† This criterion of continuity is due to Professor Taber.

determinant Δ corresponding to this structure is $\Delta = \dfrac{e^{a_2} - 1}{a_2}$, and this vanishes for a_2 an even multiple, not zero, of $\pi\sqrt{-1}$. Nevertheless the group $G_2^{(1)}$ is continuous, whereas the group $G_2^{(2)}$ is discontinuous.

The symbolic equation $e^{a_1} = e^{a + \delta t \gamma}$ is equivalent to the system of equations

$$a_k^{(1)} = a_k + \delta t\, c_k$$
$$(k = 1, 2 \ldots r),$$

which define the infinitesimal transformation of the parameter group. But the infinitesimal transformation of the parameter group is defined by the equations

$$a_k^{(1)} = a_k + \sum_1^r {}_j\, \xi_{kj}(a)\, b_j\, \delta t$$
$$(k = 1, 2 \ldots r).^*$$

Therefore

$$c_k = \sum_1^r {}_j\, \xi_{kj}(a)\, b_j$$
$$(k = 1, 2 \ldots r).$$

If $\Delta \neq 0$, equations (28) give

$$c_k = \sum_1^r {}_j \frac{A_{kj}}{\Delta} b_j$$
$$(k = 1, 2 \ldots r).$$

Therefore, if $\mathfrak{A}_1 \ldots \mathfrak{A}_r$ denote the symbols of infinitesimal transformation of the parameter group, we have

$$\mathfrak{A}_j = \sum_1^r {}_k\, \xi_{jk}(a) \frac{\partial}{\partial a_k} = \sum_1^r {}_k \frac{A_{jk}}{\Delta} \frac{\partial}{\partial a_k}$$
$$(j = 1, 2 \ldots r).\dagger$$

To illustrate what precedes, consider the two-parameter structure

$$(X_1, X_2) \equiv X_1.$$

Equation (26) gives

$$b_1 X_1 + b_2 X_2 = c_1 X_1 + c_2 X_2 - \frac{1}{2!}(a_1 c_2 - a_2 c_1) X_1 - \frac{a_2}{3!}(a_1 c_2 - a_2 c_1) X_1$$
$$- \frac{a_2^2}{4!}(a_1 c_2 - a_2 c_1) X_1 - \frac{a_2^3}{5!}(a_1 c_2 - a_2 c_1) X_1 - \ldots$$

whence follows

$$(29) \qquad b_1 = \frac{e^{a_2} - 1}{a_2} c_1 - \frac{a_1}{a_2^2}(e^{a_2} - a_2 - 1)\, c_2, \qquad b_2 = c_2.$$

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* Transformationsgruppen, I. 55, 65.

† Engel and Schur, Transformationsgruppen, III. 754 *et seq.* and 788 *et seq.*

Consequently

$$\Delta = \begin{vmatrix} \frac{e^{a_2}-1}{a_2}, & -\frac{a_1}{a_2^2}(e^{a_2}-a_2-1) \\ 0, & 1 \end{vmatrix}$$

For a_2 an even multiple, not zero, of $\pi\sqrt{-1}$, Δ vanishes. Thus it is possible that some group of the above structure shall be discontinuous.

Equations (29) give

(30)
$$c_1 = \frac{a_2}{e^{a_2}-1}\left[b_1 + \frac{a_1}{a_2^2}(e^{a_2}-a_2-1)b_2\right] \equiv \sum_1^r {}_j\, \xi_{1j}(a)\, b_j,$$
$$c_2 = b_2 \qquad \equiv \sum_1^r {}_j\, \xi_{2j}(a)\, b_j.$$

Therefore the infinitesimal transformations of the parameter group are

$$\mathfrak{A}_1 \equiv \frac{a_2}{e^{a_2}-1}\frac{\partial}{\partial a_1}.$$

$$\mathfrak{A}_2 \equiv \frac{a_1(e^{a_2}-a_2-1)}{a_2(e^{a_2}-1)}\frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_2}.$$

As a second example, take the three-parameter structure

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv X_1.$$

Equation (26) gives

$$b_1 X_1 + b_2 X_2 + b_3 X_3 = c_1 X_1 + c_2 X_2 + c_3 X_3 - \tfrac{1}{2}(a_2 c_3 - a_3 c_2) X_1.$$

Therefore

(31)
$$b_1 = c_1 + \frac{a_3}{2}c_2 - \frac{a_2}{2}c_3,$$
$$b_2 = c_2,$$
$$b_3 = c_3;$$

and

$$\Delta = \begin{vmatrix} 1, & \frac{a_3}{2}, & -\frac{a_2}{2} \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{vmatrix} = 1.$$

Whence it follows that all groups of the above structure are continuous.

Equations (31) give

$$
(32) \qquad
\begin{aligned}
c_1 &= b_1 - \frac{a_3}{2} b_2 + \frac{a_2}{2} b_3 \equiv \sum_1^r{}_j \, \xi_{1j}(a)\, b_j, \\
c_2 &= b_2 \qquad\qquad\qquad\quad\; \equiv \sum_1^r{}_j \, \xi_{2j}(a)\, b_j, \\
c_3 &= b_3 \qquad\qquad\qquad\quad\; \equiv \sum_1^r{}_j \, \xi_{3j}(a)\, b_j.
\end{aligned}
$$

Therefore, the infinitesimal transformations of the parameter group are

$$
\begin{aligned}
\mathfrak{A}_1 &\equiv \frac{\partial}{\partial a_1}, \\
\mathfrak{A}_2 &\equiv -\frac{a_3}{2}\frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_2}, \\
\mathfrak{A}_3 &\equiv \frac{a_2}{2}\frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_3}.
\end{aligned}
$$

By means of the methods explained above, I have examined the determinant Δ, and the adjoined group, corresponding to all possible types of structure of two-, three-, and four-parameter complex groups,* and the results are summarized in the table on pages 591–597, Vol. XXXV. of These Proceedings. For all types considered, the several elements (a, γ), $(a, (a, \gamma))$, etc., of (26) were calculated, and the Δ determined by the actual summation of the resulting series. Since making these calculations, Professor Taber has discovered a method of obtaining Δ immediately from (a, γ); † namely, we have

$$
(a, \gamma) = \sum_1{}_j \left(\begin{vmatrix} a_1, c_1 \\ a_2, c_2 \end{vmatrix} c_{12j} + \begin{vmatrix} a_1, c_1 \\ a_3, c_3 \end{vmatrix} c_{13j} + \ldots \right) X_j .
$$

Let, now, ϕ denote the matrix

$$
\begin{pmatrix}
-\Sigma a_j c_{j11}, & -\Sigma a_j c_{j21} & \ldots & -\Sigma_j a_j c_{jr1} \\
-\Sigma a_j c_{j12}, & -\Sigma a_j c_{j22} & \ldots & -\Sigma_j a_j c_{jr2} \\
\ldots & \ldots & \ldots & \ldots \\
-\Sigma a_j c_{j1r}, & -\Sigma a_j c_{j2r} & \ldots & -\Sigma_j a_j c_{jrr}
\end{pmatrix}
$$

Then Δ is the determinant of the matrix $\dfrac{e^{\phi} - 1}{\phi}$; that is, if $\rho_1 \ldots \rho_r$ are the roots of the characteristic equation of ϕ, $\Delta = \prod_1^r{}_j \left(\dfrac{e^{\rho_j} - 1}{\rho_j} \right)$. The

* These structures are enumerated by Lie on pp. 565, 571, 574–589, Continuierliche Gruppen; and also on pp. 713, 716, 723–730, Transformationsgruppen, III.

† This method had previously been given by Professor Engel. See Transformationsgruppen, III. 788.

constituents of Δ are integral functions of the constituents of ϕ, and therefore integral functions of $a_1 \ldots a_r$.*

In every case for which the determinant Δ vanishes for certain systems of values of $a_1 \ldots a_r$, I have found at least one group of the corresponding structure which is discontinuous.

I have also determined the infinitesimal transformations which generate the parameter group corresponding to each structure enumerated in the above mentioned table; but since the symbols are in many cases very complicated, and are of no especial interest in themselves, I have not given them.

§ 4.

In this section let the variables and parameters be restricted to real values. We will then consider the continuity of real groups, that is, groups all of whose transformations are real.

Let
$$x'_i = f_i(x_1 \ldots x_n, a_1 \ldots a_r)$$
$$(i = 1, 2 \ldots n),$$

in which the f's are analytic functions of their arguments, define an r-parameter group of real transformations. Lie's chief theorem then states that r real, linearly independent, infinitesimal transformations

$$X_j \equiv \sum_1^n{}_k \, \xi_{jk}(x) \frac{\partial}{\partial x_k}$$
$$(j = 1, 2 \ldots r)$$

in the n real variables $x_1 \ldots x_n$, generate an r-parameter real group G_r if and only if $X_1 \ldots X_r$ satisfy the conditions

$$(X_j, X_k) \equiv \sum_1^r{}_s \, c_{jks} X_s$$
$$(j, k = 1, 2 \ldots r),$$

where the c_{jks} are real quantities independent of the X's.†

Since the structural constants c_{jks} must be real, there are more types of structure possible for real groups than for complex groups. For example, for the three-parameter structures

$$(X_1, X_2) \equiv X_1, (X_1, X_3) \equiv 2 X_2, (X_2, X_3) \equiv X_3,$$
and
$$(X_1, X_2) \equiv -2 X_1, (X_1, X_3) \equiv X_2, (X_2, X_3) \equiv -2 X_3,$$

* Taber: These Proceedings, XXXV. 581.

† Transformationsgruppen, III. 360 *et seq.*

the structural constants $c_{\rho\mu s}$ are real, and one of these structures can be transformed into the other, but only by means of an imaginary transformation; consequently these structures are distinct for real groups.

The only possible types of structure of real or complex two-parameter groups are $(X_1, X_2) \equiv X_1$, and $(X_1, X_2) \equiv 0$. For the structure $(X_1, X_2) \equiv X_1$, $\Delta = \dfrac{e^{a_2} - 1}{a_2}$, which does not vanish for any real system of values of a_1, a_2; consequently all real groups of this structure are continuous. For the structure $(X_1, X_2) \equiv 0$, $\Delta = 1$; consequently all real and complex groups of this structure are continuous. Therefore all two-parameter real groups are continuous.

However, there exist three-parameter real groups which are discontinuous. Thus, let

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv X_2, \quad (X_2, X_3) \equiv - X_1.$$

For this structure we have

$$\Delta = \frac{e^{a_3\sqrt{-1}} - 1}{a_3\sqrt{-1}} \cdot \frac{e^{-a_3\sqrt{-1}} - 1}{-a_3\sqrt{-1}},$$

and Δ vanishes for real values of a_3, namely, when a_3 is an even multiple, not zero, of π. This indicates the possibility that discontinuous real groups of this structure may exist. The theorem in relation to the adjoined group, given in § 2, holds true also for real groups; namely, if the adjoined of a given real group G_r is discontinuous, G_r itself, and all groups having the same structure as G_r, are discontinuous. The adjoined group corresponding to the above structure is, however, continuous, and consequently not every group of this structure is necessarily discontinuous. Nevertheless, the group p_1, p_2, $x_1 p_2 - x_2 p_1 + p_3$, of the above structure, is discontinuous.* Its finite equations in the canonical form are

$$x'_1 = \frac{x_1}{2}\left(e^{a_3\sqrt{-1}} + e^{-a_3\sqrt{-1}}\right) - \frac{a_1\sqrt{-1}}{2\,a_3}\left(e^{a_3\sqrt{-1}} - e^{-a_3\sqrt{-1}}\right)$$
$$+ \frac{x_2\sqrt{-1}}{2}\left(e^{a_3\sqrt{-1}} - e^{-a_3\sqrt{-1}}\right) + \frac{a_2}{2\,a_3}\left(e^{a_3\sqrt{-1}} + e^{-a_3\sqrt{-1}} - 2\right),$$

$$x'_2 = \frac{x_2}{2}\left(e^{a_3\sqrt{-1}} + e^{-a_3\sqrt{-1}}\right) - \frac{a_2\sqrt{-1}}{2\,a_3}\left(e^{a_3\sqrt{-1}} - e^{-a_3\sqrt{-1}}\right)$$
$$- \frac{x_1\sqrt{-1}}{2}\left(e^{a_3\sqrt{-1}} - e^{-a_3\sqrt{-1}}\right) - \frac{a_1}{2\,a_3}\left(e^{a_3\sqrt{-1}} + e^{-a_3\sqrt{-1}} - 2\right),$$

$$x'_3 = x_3 + a_3.$$

* This is one of the real groups of Euclidean movement in three dimensional space. Cf. Transformationsgruppen, III. 385.

If this transformation is denoted by T_a, then from the symbolic equation $T_b\,T_a = T_c$ we obtain the five relations

$$(27)\quad e^{c_3\sqrt{-1}} + e^{-c_3\sqrt{-1}} = \tfrac{1}{2}\,(e^{b_3\sqrt{-1}} + e^{-b_3\sqrt{-1}})\,(e^{a_3\sqrt{-1}} + e^{-a_3\sqrt{-1}}) + \tfrac{1}{2}\,(e^{b_3\sqrt{-1}} - e^{-b_3\sqrt{-1}})\,(e^{a_3\sqrt{-1}} - e^{-a_3\sqrt{-1}}),$$

$$(28)\quad e^{c_3\sqrt{-1}} - e^{-c_3\sqrt{-1}} = \tfrac{1}{2}\,(e^{b_3\sqrt{-1}} + e^{-b_3\sqrt{-1}})\,(e^{a_3\sqrt{-1}} - e^{-a_3\sqrt{-1}}) + \tfrac{1}{2}\,(e^{b_3\sqrt{-1}} - e^{-b_3\sqrt{-1}})\,(e^{a_3\sqrt{-1}} + e^{-a_3\sqrt{-1}}),$$

$$(29)\quad c_3 = a_3 + b_3,$$

$$\begin{aligned}(30)\quad & \frac{c_2}{2\,c_3}\,(e^{c_3\sqrt{-1}} + e^{-c_3\sqrt{-1}} - 2) - \frac{c_1\sqrt{-1}}{2\,c_3}\,(e^{c_3\sqrt{-1}} - e^{-c_3\sqrt{-1}}) \\ & = \frac{b_2}{2\,b_3}\,(e^{b_3\sqrt{-1}} + e^{-b_3\sqrt{-1}} - 2) - \frac{b_1\sqrt{-1}}{2\,b_3}\,(e^{b_3\sqrt{-1}} - e^{-b_3\sqrt{-1}}) \\ & + \tfrac{1}{2}\,(e^{b_3\sqrt{-1}} + e^{-b_3\sqrt{-1}}) \\ & \times \left\{\frac{a_2}{2\,a_3}\,(e^{a_3\sqrt{-1}} + e^{-a_3\sqrt{-1}} - 2) - \frac{a_1\sqrt{-1}}{2\,a_3}\,(e^{a_3\sqrt{-1}} - e^{-a_3\sqrt{-1}})\right\} \\ & + \frac{\sqrt{-1}}{2}\,(e^{b_3\sqrt{-1}} - e^{-b_3\sqrt{-1}}) \\ \times & \left\{-\frac{a_2\sqrt{-1}}{2\,a_3}(e^{a_3\sqrt{-1}} - e^{-a_3\sqrt{-1}}) - \frac{a_1}{2a_3}\,(e^{a_3\sqrt{-1}} + e^{-a_3\sqrt{-1}} - 2)\right\} \equiv \chi,\end{aligned}$$

$$\begin{aligned}(31)\quad & \frac{c_1}{2\,c_3}\,(e^{c_3\sqrt{-1}} + e^{-c_3\sqrt{-1}} - 2) + \frac{c_2\sqrt{-1}}{2\,c_3}\,(e^{c_3\sqrt{-1}} - e^{-c_3\sqrt{-1}}) \\ & = \frac{b_1}{2\,b_3}\,(e^{b_3\sqrt{-1}} + e^{-b_3\sqrt{-1}} - 2) + \frac{b_2\sqrt{-1}}{2\,b_3}\,(e^{b_3\sqrt{-1}} - e^{-b_3\sqrt{-1}}) \\ & + \tfrac{1}{2}\,(e^{b_3\sqrt{-1}} + e^{-b_3\sqrt{-1}}) \\ & \times \left\{\frac{a_1}{2a_3}\,(e^{a_3\sqrt{-1}} + e^{-a_3\sqrt{-1}} - 2) + \frac{a_2\sqrt{-1}}{2\,a_3}\,(e^{a_3\sqrt{-1}} - e^{-a_3\sqrt{-1}})\right\} \\ & + \frac{\sqrt{-1}}{2}\,(e^{b_3\sqrt{-1}} - e^{-b_3\sqrt{-1}}) \\ & \times \left\{\frac{a_2}{2a_3}(e^{a_3\sqrt{-1}} + e^{-a_3\sqrt{-1}} - 2) - \frac{a_1\sqrt{-1}}{2\,a_3}(e^{a_3\sqrt{-1}} - e^{-a_3\sqrt{-1}})\right\} \equiv \psi.\end{aligned}$$

Denoting the right-hand members of equations (30) and (31) by χ and ψ respectively, and solving for c_1, c_2, c_3, we have

$$
\begin{aligned}
& c_1 = \frac{a_3 + b_3}{2} \left\{ \frac{\sqrt{-1}\,(e^{\frac{1}{2}(a_3+b_3)\sqrt{-1}} + e^{-\frac{1}{2}(a_3+b_3)\sqrt{-1}})}{(e^{\frac{1}{2}(a_3+b_3)\sqrt{-1}} - e^{-\frac{1}{2}(a_3+b_3)\sqrt{-1}})}\,\chi - \psi \right\}, \\
(32)\quad & c_2 = -\frac{a_3 + b_3}{2} \left\{ \chi - \sqrt{-1}\,\psi \frac{(e^{\frac{1}{2}(a_3+b_3)\sqrt{-1}} + e^{-\frac{1}{2}(a_3+b_3)\sqrt{-1}})}{(e^{\frac{1}{2}(a_3+b_3)\sqrt{-1}} - e^{-\frac{1}{2}(a_3+b_3)\sqrt{-1}})} \right\}, \\
& c_3 = a_3 + b_3.
\end{aligned}
$$

If the a's and b's are so chosen that χ and ψ are different from zero, and $a_3 + b_3 = 4\,k\,\pi$, where k is an arbitrary integer, both c_1 and c_2 become infinite. Consequently this group is discontinuous.

On pages 106–107, 384, Vol. III., Transformationsgruppen, Lie enumerates all possible types of real projective groups of the plane. I have examined all the two-, three-, and four-parameter groups in this list, and find that the groups

$$x_1 p_2, \quad x_1 p_1 - x_2 p_2, \quad x_2 p_1,$$

and

$$p_1 + x^2{}_1 p_1 + x_1 x_2 p_2, \quad p_2 + x_1 x_2 p_1 + x^2{}_2 p_2, \quad x_2 p_1 - x_1 p_2,$$

and these only, are discontinuous.

The first of these groups is the special linear homogeneous real group, and has the structure

$$(X_1, X_2) \equiv -2\,X_1,\ (X_1, X_3) \equiv X_2,\ (X_2, X_3) \equiv -2\,X_3.$$

The determinant Δ corresponding to this structure is

$$\Delta = \frac{e^{2\sqrt{a^2{}_2 + a_1 a_3}} - 1}{2\sqrt{a^2{}_2 + a_1 a_3}} \cdot \frac{e^{-2\sqrt{a^2{}_2 + a_1 a_3}} - 1}{-2\sqrt{a^2{}_2 + a_1 a_3}}.$$

This vanishes if the a's are so chosen as to satisfy the condition

$$a^2{}_2 + a_1 a_3 = -\,k^2 \pi^2,$$

where k is an arbitrary integer.*

The second of the above groups has the structure

$$(X_1, X_2) \equiv X_3, \quad (X_1, X_3) \equiv -X_2, \quad (X_2, X_3) \equiv X_1.$$

The Δ corresponding to this structure is

* The special linear homogeneous *complex* group has been shown to be discontinuous by Professor Study, Leipziger Berichte, 1892; and the *real* group by Professor Taber, Bull. Am. Math. Soc., April, 1896. The *general* linear homogeneous (real or complex) group is continuous. Thus a group may be continuous and yet have a discontinuous sub-group.

$$\Delta = \frac{e^{\sqrt{-(a^2_1+a^2_2+a^2_3)}}-1}{\sqrt{-(a^2_1+a^2_2+a^2_3)}} \cdot \frac{e^{-\sqrt{-(a^2_1+a^2_2+a^2_3)}}-1}{-\sqrt{-(a^2_1+a^2_2+a^2_3)}},$$

and this vanishes if the a's satisfy the condition $a^2_1 + a^2_2 + a^2_3 = 4\,k^2\pi^2$, where k is an arbitrary integer. The adjoined of this group is

$$a_1 \frac{\partial}{\partial a_3} - a_3 \frac{\partial}{\partial a_1}, \quad a_3 \frac{\partial}{\partial a_2} - a_2 \frac{\partial}{\partial a_3}, \quad a_2 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial a_2},$$

and is discontinuous. Consequently, every group of this type is discontinuous.

In the table on pages 391–397, Vol. XXXV., of These Proceedings, as mentioned above, I have marked by an asterisk those types of structure for which all real groups are continuous; and by a dagger those types of structure for which I have found at least one real group that is discontinuous.